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Geometric phase in a Bose-Einstein-Josephson junction

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Abstract. We calculate the geometric phase associated with the time evolution of the wave function of a Bose-Einstein condensate system in a double-well trap by using a model for tunneling between the wells. For a cyclic evolution, this phase is shown to be half the solid angle subtended by the evolution of a unit vector whose z-component and azimuthal angle are given, respectively, by the population difference and phase difference between the two condensates. For a non-cyclic evolution, an additional phase term arises. We show that the geometric phase can also be obtained by mapping the tunneling equations on to the equations of a space curve. The importance of a geometric phase in the context of some recent experiments is pointed out.

PACS. 02.40.Hw Classical differential geometry – 05.45.-a Nonlinear dynamics and nonlinear dynamical systems

1 Introduction

Bose-Einstein condensation in a dilute gas of trapped ultracold alkali atoms has been observed by several experimental groups [1]. This gives rise to the possibility of understanding the nature of the condensate wave function, and in particular, its phase [2]. It is believed that this phase transition occurs due the breaking of a global gauge symmetry of the Hamiltonian. A Bose-Einstein condensate (BEC) may be modeled by writing down the interacting many-body Hamiltonian in terms of boson creation and annihilation operators $\Psi_{\rm op}^{\dagger}$ and $\Psi_{\rm op}$. The order parameter is postulated to be the condensate wave function $\psi = \langle \Psi_{\rm op} \rangle = \rho\, e^{i\theta}$, where $\rho = |\psi|^2$ is the condensate density and θ is the phase of the wave function. The Hamiltonian is gauge invariant, but the order parameter breaks this symmetry. Using the dynamical equation for $\Psi_{\rm op}$ found from the Hamiltonian operator, the time evolution of the condensate wave function ψ can be shown to satisfy the following nonlinear Schrödinger equation, i.e., the Gross-Pitaevskii equation (GPE) [3]:

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \bigtriangledown^2 \psi + \left[V_{\rm ext}(\mathbf{x}) + g_0 |\psi|^2\right] \psi, \qquad (1.1)$$

where $V_{\rm ext}$ is the external potential and $g_0 = 4\pi\hbar^2 a/m$, a and m being the atomic scattering length and mass, respectively. Although this equation has an underlying quantum nature, the condensate has a macroscopic extent, suggesting the observation of quantum effects on a macroscopic scale.

In a striking experiment, Andrews et al. [4] have shown the existence of the macroscopic quantum phase difference between two BECs: they designed a double-well trap by using a laser sheet to create a high barrier within a trapped condensate. On *switching off* this barrier, the two condensates overlapped to produce an interference pattern, showing phase coherence. More interestingly, by *lowering* the laser sheet intensity, the barrier gets lowered, making it possible for the condensates to *tunnel* through the barrier. Thus this double-well trap is analogous to a superconductor Josephson junction [5], and is referred to as the Bose-Einstein Josephson junction (BJJ). In an interesting paper, Smerzi et al. [6] have set up the tunneling equations for the BJJ in a model. These are two coupled nonlinear ordinary differential equations for the condensate wave functions in the two wells. They have studied the time evolution of the inter-well population difference and phase difference in this model, and predicted a novel 'self-trapping' effect, i.e., the oscillation of the population difference around a non-zero value, for certain initial conditions and parameters. It must be mentioned that a similar effect had been found by Kenkre and Campbell [7] in the context of the *discrete* nonlinear Schrödinger equation.

The tunneling dynamics motivates the following question: Is there an underlying geometric phase associated with the time evolution of the condensate wave function in a double-well trap? As is well-known by now, the concept of a geometric phase has been studied in various contexts, after it was introduced by Berry [8] in quantum mechanics. It had also been considered much earlier by Pancharatnam [9] in the context of classical optics. Geometric phase and its various applications have been

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studied intensively for over a decade now [10]. Such a (non-integrable) phase arises when the time evolution of a system is such that the value of a variable in a given state of the system depends on the path along which the state has been reached. In this paper, we calculate the geometric phase associated with the time evolution of the BJJ wave function, for both cyclic and non-cyclic evolutions.

The plan of the paper is as follows: in Section 2, we briefly review the derivation of the tunneling equations by Smerzi et al. [6]. Keeping in mind that the geometric phase is gauge-independent, we use a certain gauge transformation to reduce these equations to a more convenient form. In Section 3, we outline the kinematic approach formulated by Mukunda and Simon [11] to define the geometric phase as applied to a two-level system. We then solve the BJJ tunneling equations, which are nonlinear differential equations, numerically by choosing some parameter values as an illustrative example. Using these solutions, we find the geometric phase explicitly, for both cyclic and non-cyclic evolution of the system. For a cyclic evolution, the phase difference and the population difference between the condensates in the two wells return to their original values. For this case, the corresponding geometric phase is half the solid angle generated by a *unit vector* whose z-component and azimuthal angle are given, respectively, by the population difference and the phase difference. For non-cyclic evolution, an additional phase term is obtained. In Section 4, we show that this geometric phase (for both types of evolution) can also be obtained by first mapping the tunneling equations to the equation for a unit vector $\mathbf{r}$ and then identifying it with the tangent $\mathbf{T}$ of a space curve. The space curve is described using the so-called natural frame equations, which possess an underlying natural gauge freedom. The unit triad of vectors can be written down using the form of the condensate wave functions in the two wells. The concept of Fermi-Walker parallel transport is then used to identify the geometric phase. In Section 5, we employ the usual Frenet frame to obtain explicit expressions for the curvature and torsion of the space curve that gets associated with the BJJ evolution. In Section 6, we discuss some recent experiments and summarize our results.

2 The BJJ tunneling equations

We begin by briefly describing the model used by Smerzi et al. [6] to study the tunneling of the condensate between two wells. Let the total number of atoms in the double-well trap be N. Let N_1 and N_2 denote the number of atoms in each well, such that $N_1+N_2=N$. To study the tunneling, the solution for the GPE (Eq. (1.1)) is assumed to be of the form

$$\psi = \psi_1(t)\Phi_1(\mathbf{x}) + \psi_2(t)\Phi_2(\mathbf{x}). \tag{2.1}$$

Here Φ_1, Φ_2 are the ground state solutions for the isolated wells with $N_1 = N_2 = (N/2)$. Using equation (2.1) in equation (1.1) [6,12], and using a gauge transformation of the form

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = \sqrt{N} e^{i\int \eta(t')dt'} \begin{pmatrix}a\\ b\end{pmatrix}, \tag{2.2}$$

the BJJ tunneling equations take on the form

$$i\hbar \frac{d}{dt}\begin{pmatrix}a\\ b\end{pmatrix} = \begin{pmatrix}\hbar\omega_0 & -V\\ -V & -\hbar\omega_0\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix} = M_{\omega_0}\begin{pmatrix}a\\ b\end{pmatrix}, \tag{2.3}$$

with an appropriately defined $\eta(t)$. Further, $\hbar\omega_0$ can be written as the sum of the asymmetry in the energies of the bosons in the two wells and the interaction energy between bosons in each well. V is a measure of the overlap between the wave functions in the two wells.

From equation (2.2), we see that the normalization condition $|\psi_1|^2 + |\psi_2|^2 = N$ implies $|a|^2 + |b|^2 = 1$. Thus without loss of generality, we write

$$a = \cos(\alpha/2)\, e^{i\theta_1}; \quad b = \sin(\alpha/2)\, e^{i\theta_2}. \tag{2.4}$$

Let us denote the difference in the population density of the two traps by z and the difference in the phases of the two condensates by ϕ. From equation (2.4) we thus have,

$$z = (N_1 - N_2)/N = (|a|^2 - |b|^2) = \cos\alpha; \quad \phi = (\theta_2 - \theta_1). \tag{2.5}$$

By suitably combining equation (2.3) and its complex conjugate, and using equation (2.5), the nonlinear coupled equations for z and ϕ are found to be (on setting $\hbar = 1$)

$$\frac{dz}{dt} = -V\sqrt{1-z^2}\sin\phi \tag{2.6a}$$

$$\frac{d\phi}{dt} = \Lambda z + V\frac{z}{\sqrt{1-z^2}}\cos\phi + \Delta E. \tag{2.6b}$$

Here ΔE is a measure of the asymmetry between the two wells, Λ depends on the strength of the interaction between the Bose atoms in the condensate, and the time has been reparametrized as $t \to 2t$.

It is interesting to note that the above equations can also be written as Hamilton's equations, by treating z and ϕ as the canonically conjugate variables. The classical Hamiltonian is easily verified to be

$$H_{cl} = \Lambda\frac{z^2}{2} - V\sqrt{1-z^2}\cos\phi + \Delta E z. \tag{2.7}$$

This describes a non-rigid or "momentum-shortened" pendulum, since its length is proportional to $\sqrt{1-z^2}$, which decreases with the "momentum" z. This system has been studied in detail in [12]. In [13] the effect of an extra dissipative term in the $\dot{z}$ equation has been studied. A recent review of tunneling in trapped BEC is given in [14].

At this point, a digressive remark is in order. The derivation of equation (2.3) as given here is a certain approximation to the original theory. Strictly speaking, the dynamical variables $\psi_1(t)$ and $\psi_2(t)$ must be identified with bosonic operators a_1 and a_2 satisfying $[a_1, a_1^+] = [a_2, a_2^+] = 1$, within an approach named space-mode approximation of the bosonic field operator, introduced by Milburn et al. [15]. For large total boson numbers, the mean-field limit becomes correct, thereby permitting the substitution of a_1 and a_2 by $\psi_1(t)$ and $\psi_2(t)$ respectively. Due to its semiclassical interpretation, this substitution can be effected in a consistent way by relying on

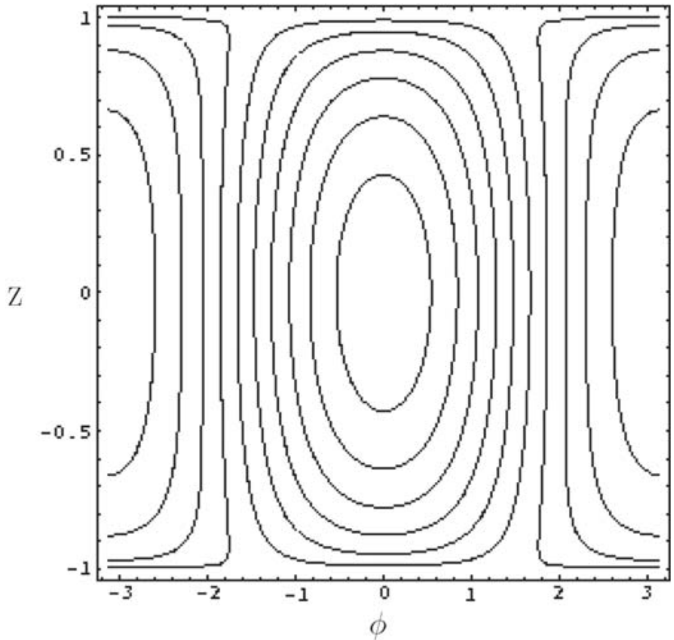

Fig. 1. Phase portrait of the BJJ evolution (Eq. (2.7)) for an interacting Bose system with $\Lambda = 0.5$, in a symmetric trap.

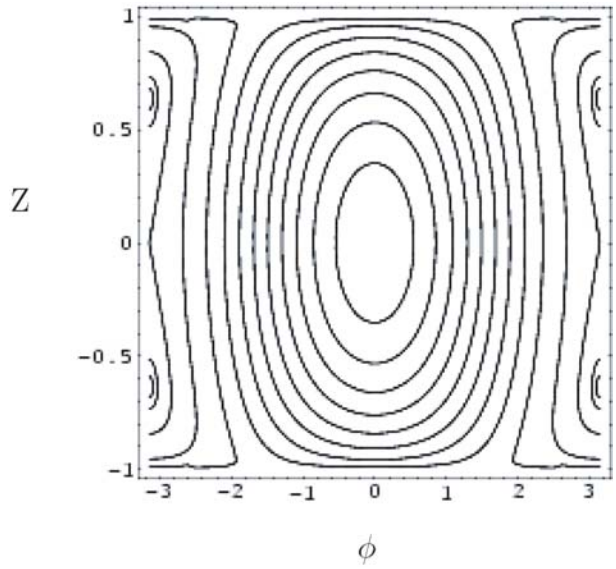

Fig. 2. Phase portrait of the BJJ evolution (Eq. (2.7)) for an interacting Bose system with $\Lambda = 1.3$, in a symmetric trap. The trapped states at $\phi = \pi$ are clearly visible here.

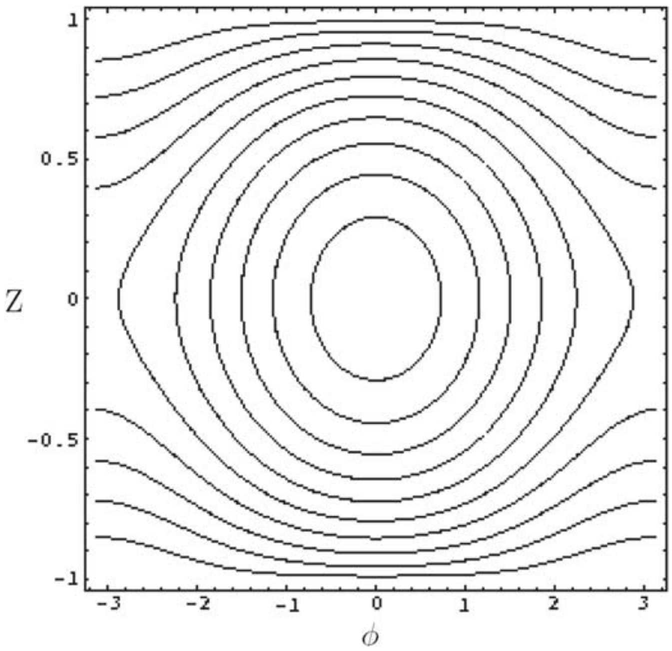

Fig. 3. Phase portrait of the BJJ evolution (Eq. (2.7)) for an interacting Bose system with $\Lambda = 5.0$, in a symmetric trap.

the coherent-state variational picture of quantum dynamics [16]. Recently, Buonsante et al. [17] have studied the phase space of the two-well system as a special subcase of a three-well system.

It is possible to write down the expression for ω_0 as

$$\omega_0 = \Delta E + \Lambda z. \tag{2.8}$$

Finally, setting $z = \cos\alpha$ in equations (2.6a) and (2.6b), we obtain

$$\frac{d\alpha}{dt} = V \sin\phi \tag{2.9a}$$

$$\frac{d\phi}{dt} = \Lambda \cos\alpha + V \cot\alpha \cos\phi + \Delta E. \tag{2.9b}$$

Equations (2.9), or equivalently, equations (2.6), represent the tunneling equations. Since the phase space plots of this Hamiltonian system have not been displayed in the literature, we include some illustrative examples here. We consider two special limits of the Hamiltonian.

(i) Interacting Bose system in a symmetric trap: here, $\Lambda \neq 0$, $\Delta E = 0$. Thus equation (2.7) becomes $H_{cl} = \Lambda\frac{z^2}{2} - V\sqrt{1-z^2}\cos\phi$. In Figures 1, 2 and 3, we have obtained the (z, ϕ) phase portraits for this case. Let Λ be replaced by the dimensionless quantity (Λ/V). For $\Lambda < 1$ there exist periodic oscillations around the zero-state $(0, 0)$ and the non-trapped π state $(0, \pi)$. There are no rotational states (see Fig. 1 for $\Lambda = 0.5$). For $\Lambda > 1$ two new trapped π-states appear at (z^*, π) and $(-z^*, \pi)$ with $z^* = \sqrt{\Lambda^2 - 1}/\Lambda$. The trapped π-states are clearly visible in Figure 2 ($\Lambda = 1.3$). As Λ increases, $z^* \to 1$. For $\Lambda > 2$, rotational states also appear as seen in Figure 3 ($\Lambda = 5$).

(ii) Non-interacting Bose system: $\Lambda = 0$. In this limit the kinetic energy term of the Hamiltonian H_{cl} (2.7) vanishes and hence the momentum-shortened pendulum analogy is not valid anymore. The equations are the same as semiclassical two level atom equations. We get $H_{cl} = -V\sqrt{1-z^2}\cos\phi + \Delta E z$. When $\Delta E = 0$, we have a symmetric trap and the behavior is rather similar to that in Figure 1, with no rotational orbits. With an increase in ΔE, rotational orbits appear, which explore the full range of ϕ. This is shown in Figure 4. The oscillations are now around ($z = -z^*, \phi = 0$) and ($z = +z^*, \phi = \pm\pi$). That is, oscillations around $\phi = 0$ shift towards $z = -1$ which is energetically more stable, while the $\phi = \pi$ fixed

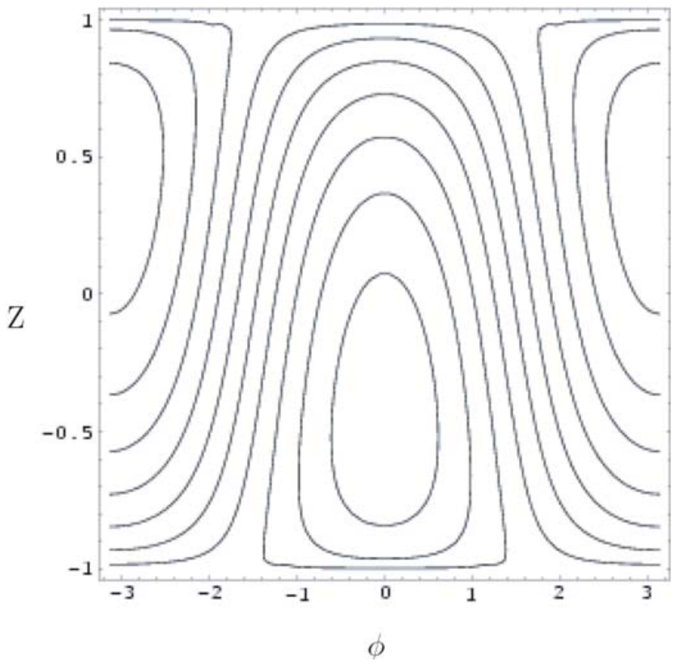

Fig. 4. Phase portrait of the BJJ evolution (Eq. (2.7)) for a non-interacting Bose system in an asymmetric trap with $\Delta E = 0.5$.

point moves towards $z = 1$ which is a local energy maximum. Note that the increase in ΔE causes a non-trapped π state to become a trapped π state with oscillations around a non-zero population difference z^*. The new fixed point is given by $z^* = \sqrt{((\Delta E/V)^2/(1+(\Delta E/V)^2))}$ which tends to 1 as $\Delta E/V$ goes to infinity.

We now proceed to show how the geometric phase associated with the BJJ dynamics can be computed.

3 Geometric phase using the kinematic approach

In this section we derive the expression for the geometric phase for the BJJ evolution using the kinematic approach developed by Mukunda and Simon [11]. Invoking the principle of gauge invariance, they obtain the geometric phase as the difference between the total phase and the dynamical phase as follows:

$$\phi_g = \arg(\psi(0), \psi(T)) - \operatorname{Im} \int_0^T dt \quad (\psi(t), \dot{\psi}(t)). \tag{3.1}$$

The first term in equation (3.1) is easily identified as the total phase

$$\phi_p = \arg(\psi(0), \psi(T)), \tag{3.2a}$$

while the second term in equation (3.1) is the dynamical phase

$$\phi_d = \operatorname{Im} \int_0^T dt \quad (\psi(t), \dot{\psi}(t)). \tag{3.2b}$$

Let us now calculate expression for the geometric phase for the BJJ evolution equations. The family of unit vectors is given by,

$$\psi = \begin{pmatrix} a \\ b \end{pmatrix} = e^{i\theta_1(t)} \begin{pmatrix} \cos\frac{1}{2}\alpha(t) \\ \sin\frac{1}{2}\alpha(t)\ e^{i\phi(t)} \end{pmatrix} = e^{i\theta_1(t)}\psi, \tag{3.3}$$

where $\phi = (\theta_2 - \theta_1)$. Using equation (3.3) in equation (3.2a), a short calculation leads to the following total phase:

$$\phi_p = \arg(\psi(0), \psi(T)) = (\theta_1(T) - \theta_1(0)) + \Delta. \tag{3.4}$$

Here,

$$\Delta = \tan^{-1}[\mu/\nu], \tag{3.5}$$

where

$$\mu = \sin(\alpha(0)/2)\sin(\alpha(T)/2)\sin(\phi(T)-\phi(0))$$

and

$$\nu = \cos(\alpha(0)/2)\cos(\alpha(T)/2) + \sin(\alpha(0)/2)\sin(\alpha(T)/2)\cos(\phi(T)-\phi(0)).$$

The integrand of the dynamical phase ϕ_d can be calculated using equation (3.3) in equation (3.2b) to give

$$\operatorname{Im}\left(\psi, \frac{d\psi}{dt}\right) = \dot{\theta}_1 + \sin^2(\alpha/2)\dot{\phi}. \tag{3.6}$$

Thus

$$\phi_d = (\theta_1(T) - \theta_1(0)) + \int_0^T \sin^2(\alpha/2)\dot{\phi}\ dt. \tag{3.7a}$$

From equations (3.4) and (3.7a), we get the geometric phase to be

$$\phi_g = \phi_p - \phi_d = -\int_0^T \sin^2(\alpha/2)\dot{\phi}\ dt + \Delta. \tag{3.7b}$$

For a cyclic evolution, it is clear from (3.7b) that $\Delta = 0$. Hence the geometric phase is just minus half the solid angle Ω subtended by the closed curve generated on a sphere by the tip of a unit vector $\mathbf{r}$:

$$\mathbf{r} = (\sin\alpha\cos\phi, \sin\alpha\sin\phi, \cos\alpha). \tag{3.8}$$

Here, α and ϕ denote the polar and azimuthal angles of $\mathbf{r}$.

An example

As an example we consider an interacting Bose system with $\Lambda = 5$, in a symmetric trap, i.e., $\Delta E = 0$, whose phase space portrait is given in Figure 3. The value of Λ selected is quite generic because further increase in Λ does not change the character of the phase-space portrait much, apart from moving the π-state towards $z = 1$. Since $\cos\alpha = z$, the geometric phase ϕ_g given in equation (3.7b) can be re-expressed as

$$\phi_g = \frac{1}{2}\int_0^T (z-1)\dot{\phi}\ dt + \Delta. \tag{3.9}$$

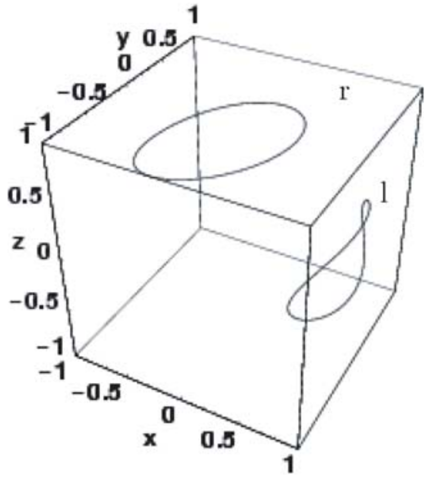

Fig. 5. The BJJ evolution of the unit vector $\mathbf{r}$ (see Eq. (4.2)) on the unit sphere: Paths corresponding to a librational orbit and a rotational orbit (labeled r and l respectively in the plot) in the phase space portrait of the BJJ Hamiltonian for a symmetric trap with $\Lambda = 5.0$ (see Fig. 3) are shown.

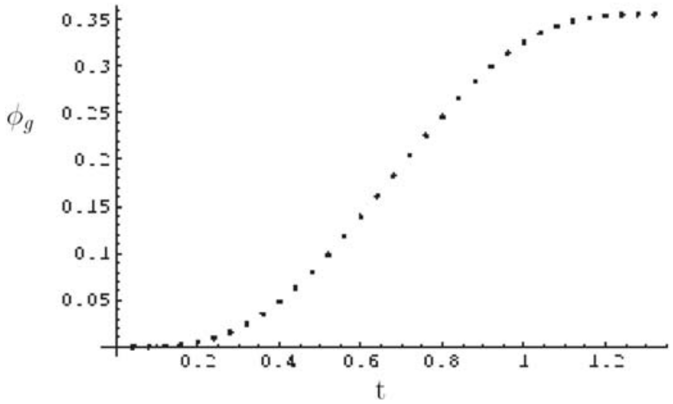

Fig. 6. Evolution of the geometric phase ϕ_g as a function of time over a period for a librational orbit (oscillation about the zero-state) at $\Lambda = 5.0$, with initial conditions $(z, \phi) = (0.3,\ 0)$ corresponding to orbit l in Figure 5.

While computing the geometric phase from the above equation, it is necessary to keep in mind that in equations (2.6), the time has been reparametrized, and so one has to use the appropriate value of time in equation (3.9). We solve equations (2.6) numerically for $(z(t), \phi(t))$, for a given initial condition $(z(0), \phi(0))$ at time $t = 0$. Using these solutions, we find the solutions for the corresponding unit vectors $\mathbf{r}$ given in equation (3.8). These yield the the path on the unit sphere plotted in Figure 5. Next, we substitute the solution for $(z(t), \phi(t))$ in equation (3.9), to find $\phi_g(t)$ numerically, for various times t, in the range $0 \le t \le T$, where T denotes the full period of the orbit concerned. Our plot for the time dependence of $\phi_g(t)$ over a period for a librational orbit is given in Figure 6, while Figure 7 gives that plot for a rotational orbit.

We conclude this section with the following remark. There exists an interesting geometrical representation of a two level system in terms of the time evolution of a

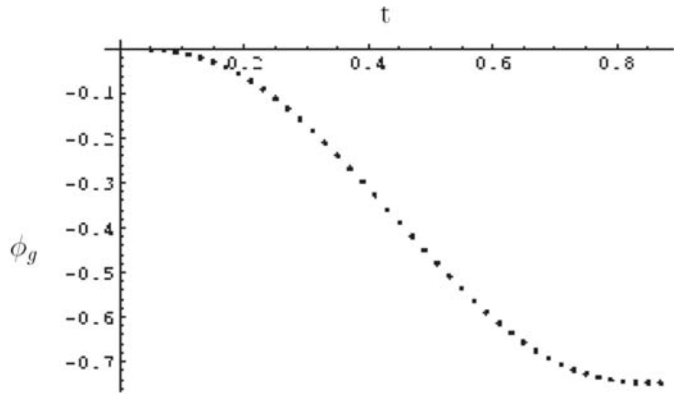

Fig. 7. Evolution of the geometric phase ϕ_g as a function of time over a period for a rotational orbit at $\Lambda = 5.0$ with initial conditions $(z, \phi) = (0.9,\ 0)$ corresponding to orbit r in Figure 5.

unit vector $\mathbf{r}$. In the next section, we identify $\mathbf{r}$ with the *tangent* of a space curve, and provide a classical differential geometric approach to derive the geometric phase ϕ_g associated with the BJJ evolution.

4 Geometric phase using space curve approach

In this section, we derive the geometric phase associated with the BEC tunneling dynamics by providing a geometric visualisation of this two level system. Firstly it is possible to show [18] that the tunneling equations for the two-level wave function (3.3), which we rewrite below for convenience as

$$\psi = \begin{pmatrix} a \\ b \end{pmatrix} = e^{i\theta_1} \begin{pmatrix} \cos(\alpha/2) \\ \sin(\alpha/2)\ e^{i\phi} \end{pmatrix}, \tag{4.1}$$

can be mapped to the following vector evolution equation

$$d\mathbf{r}/dt = \boldsymbol{\omega} \times \mathbf{r}. \tag{4.2}$$

Here, in Cartesian coordinates,

$$\boldsymbol{\omega} = (-2V, 0, 2\omega_0) \tag{4.3}$$

$$\mathbf{r} = (a^*b + ab^*, i(ab^* - a^*b), |a|^2 - |b|^2). \tag{4.4}$$

Using the definitions a and b given in equation (2.4), equation (4.4) is readily seen to be identical to the unit vector $\mathbf{r}$ in equation (3.8).

While discussing two-level systems, *adiabatic, cyclic* evolutions and Berry's phase, Urbantke [19] has shown that given a unit vector of the form (4.4), two more unit vectors $\mathbf{P}'$, $\mathbf{Q}'$ can be defined, such that the set $(\mathbf{r}, \mathbf{P}', \mathbf{Q}')$ forms a unit orthogonal right-handed triad. This is achieved by defining a complex vector $\mathbf{Z}'$ as follows:

$$\mathbf{Z}' = \mathbf{P}' + i\mathbf{Q}' = ((a^2 - b^2), i(a^2 + b^2), -2ab). \tag{4.5}$$

On using definitions of a and b in equation (4.5), we get,

$$\mathbf{Z}' = e^{2i\theta_1}\big(\cos^2(\alpha/2) - \sin^2(\alpha/2)e^{2i\phi},\\ i\big(\cos^2(\alpha/2) + \sin^2(\alpha/2)e^{2i\phi}\big), -\sin(\alpha)e^{i\phi}\big). \quad (4.6)$$

By writing down the real and imaginary parts of equation (4.6), it can be easily verified that,

$$|\mathbf{r}| = |\mathbf{P}'| = |\mathbf{Q}'| = 1; \;\; \mathbf{r}\cdot\mathbf{P}' = \mathbf{r}\cdot\mathbf{Q}' = \mathbf{P}'\cdot\mathbf{Q}' = 0. \quad (4.7)$$

Clearly, as $\mathbf{r}$ evolves with time, so does the $(\mathbf{P}', \mathbf{Q}')$ plane. We find the geometric phase as follows.

Firstly, as is obvious, the *total phase* Γ_p accumulated by $\mathbf{Z}'(t)$ in time T is given by,

$$\Gamma_p = \arg(\mathbf{Z}'(0)^* \cdot \mathbf{Z}'(T)). \quad (4.8)$$

Substituting for $\mathbf{Z}'$ from equation (4.6) into equation (4.8), after some algebra we obtain,

$$\Gamma_p = 2[(\theta_1(T) - \theta_1(0)) + \Delta]. \quad (4.9)$$

A comparison with the expression for the total phase in Section 3 shows that

$$\Gamma_p = 2\phi_p \quad (4.10)$$

ϕ_p being the total phase of ψ in the kinematic approach.

Next, we find the *total phase* rotation γ_p associated with the rotation of $\mathbf{P}'$ or $(\mathbf{Q}')$ as follows. It is defined by

$$\cos\gamma_p = \mathbf{P}'(T)\cdot\mathbf{P}'(0) = \mathbf{Q}'(T)\cdot\mathbf{Q}'(0).$$

Further, it is easy to see geometrically that $\mathbf{P}'(T)\cdot\mathbf{Q}'(0) = -\mathbf{Q}'(T)\cdot\mathbf{P}'(0) = \sin\gamma_p$. Substituting $\mathbf{Z}' = \mathbf{P}' + i\mathbf{Q}'$ in equation (4.8) and using the above relations, we can show that the total phase

$$\gamma_p = -\Gamma_p = -2[(\theta_1(T) - \theta_1(0)) + \Delta], \quad (4.11)$$

where we have used equation (4.9).

Next we wish to find the *dynamical phase* γ_d associated with $(\mathbf{P}', \mathbf{Q}')$ rotation, which is induced by the specific dynamical equations of the frame $(\mathbf{r}, \mathbf{P}', \mathbf{Q}')$. This is a little more involved, and we proceed as follows.

From equation (4.6), we have

$$\mathbf{Z}' = e^{2i\theta_1}\mathbf{Z}. \quad (4.12)$$

This immediately leads to

$$\mathbf{P}' + i\mathbf{Q}' = e^{2i\theta_1}(\mathbf{P} + i\mathbf{Q}). \quad (4.13)$$

Comparing this with equation (4.6) yields

$$\mathbf{P} = (\cos^2(\alpha/2) - \sin^2(\alpha/2)\cos 2\phi, -\sin^2(\alpha/2)\sin 2\phi,\\ -\sin\alpha\cos\phi), \quad (4.14a)$$

$$\mathbf{Q} = (-\sin^2(\alpha/2)\sin 2\phi, \cos^2(\alpha/2) + \sin^2(\alpha/2)\cos 2\phi,\\ -\sin\alpha\sin\phi). \quad (4.14b)$$

It can be easily verified that $(\mathbf{r}, \mathbf{P}, \mathbf{Q})$ is also a right-handed triad.

A short calculation using equations (3.8) and (4.14) shows that we can write

$$d\mathbf{r}/dt = X\mathbf{P} + Y\mathbf{Q}, \quad (4.15)$$

where

$$X = \left(\frac{d\alpha}{dt}\right)\cos\phi - \left(\sin\alpha\frac{d\phi}{dt}\right)\sin\phi, \quad (4.16a)$$

$$Y = \left(\sin\alpha\frac{d\phi}{dt}\right)\cos\phi + \left(\frac{d\alpha}{dt}\right)\sin\phi. \quad (4.16b)$$

Obviously, there is a gauge freedom $2\theta_1$ in the choice of $(\mathbf{P}, \mathbf{Q})$. We immediately see this from equation (4.13):

$$\mathbf{P}' = \mathbf{P}\cos\beta - \mathbf{Q}\sin\beta \quad (4.17a)$$

$$\mathbf{Q}' = \mathbf{P}\sin\beta + \mathbf{Q}\cos\beta, \quad (4.17b)$$

where

$$\beta = 2\theta_1 \quad (4.18)$$

represents the gauge freedom. Using equations (4.17), we solve for $(\mathbf{P}, \mathbf{Q})$ in terms of $(\mathbf{P}', \mathbf{Q}')$. Substituting them in equation (4.15) yields

$$\frac{d\mathbf{r}}{dt} = \alpha_1\mathbf{P}' + \alpha_2\mathbf{Q}' \quad (4.19)$$

where

$$\alpha_1 = \frac{d\alpha}{dt}\cos(\phi+\beta) - \left(\sin\alpha\frac{d\phi}{dt}\right)\sin(\phi+\beta), \quad (4.20a)$$

$$\alpha_2 = \frac{d\alpha}{dt}\sin(\phi+\beta) + \left(\sin\alpha\frac{d\phi}{dt}\right)\cos(\phi+\beta). \quad (4.20b)$$

Since $(\mathbf{r}, \mathbf{P}', \mathbf{Q}')$ is an orthonormal triad, equation (4.19) immediately implies,

$$\frac{d\mathbf{P}'}{dt} = -\alpha_1\mathbf{r} + \alpha_3\mathbf{Q}' \quad (4.21)$$

$$\frac{d\mathbf{Q}'}{dt} = -\alpha_2\mathbf{r} - \alpha_3\mathbf{P}' \quad (4.22)$$

where α_3 is to be determined. In the space curve language, if $\mathbf{r}$ is identified with the tangent $\mathbf{T}$, then α_1 and α_2 are the components of the curvature vector $d\mathbf{T}/dt$ along $\mathbf{P}'$ and $\mathbf{Q}'$ respectively. Further, equations (4.19), (4.21) and (4.22) describe the equations for a space curve in a "natural frame" $(\mathbf{T}, \mathbf{P}', \mathbf{Q}')$. We remark that the Frenet frame [20] corresponds to $\alpha_2 = 0$, $\mathbf{P}'$ is the normal $\mathbf{n}$, $\mathbf{Q}'$ is the binormal $\mathbf{b}$. Further, α_3 is the torsion τ and α_1 is the curvature K. On setting $\alpha_2 = 0$, we get from equation (4.20), the following "Frenet gauge" β_F:

$$\tan(\beta_F + \phi) = \sin\alpha\frac{d\phi}{dt}\bigg/\left(\frac{d\alpha}{dt}\right). \quad (4.23)$$

Working with the natural frame, a short calculation using equations (4.15) to (4.18) yields,

$$\alpha_3 = \mathbf{T}\cdot(\dot{\mathbf{T}}\times\ddot{\mathbf{T}})/|\dot{\mathbf{T}}|^2 - \frac{d}{dt}\tan^{-1}\left(\frac{\alpha_2}{\alpha_1}\right). \tag{4.24}$$

Next using the Cartesian representation of $\mathbf{T}=\mathbf{r}$ given in equation (4.4), a lengthy but straightforward calculation leads to,

$$\mathbf{T}\cdot(\dot{\mathbf{T}}\times\ddot{\mathbf{T}})/|\dot{\mathbf{T}}|^2 = \cos\alpha\frac{d\phi}{dt}+\frac{d}{dt}\tan^{-1}\left[\frac{\sin\alpha\frac{d\phi}{dt}}{\frac{d\alpha}{dt}}\right]. \tag{4.25}$$

Substituting equation (4.25) and (4.20) in equation (4.24) and using the formula $\tan^{-1}A-\tan^{-1}B=\tan^{-1}((A-B)/(1+AB))$, we obtain,

$$\alpha_3 = \cos\alpha\frac{d\phi}{dt}-\frac{d(\phi+\beta)}{dt} = -2\sin^2\frac{\alpha}{2}\frac{d\phi}{dt}-\frac{d\beta}{dt}. \tag{4.26}$$

Note that the time derivative of the gauge freedom $\beta(t)$ appears in α_3.

We write equations (4.19), (4.21) and (4.22) in a compact form,

$$\frac{d\mathbf{T}}{dt}=\boldsymbol{\xi}\times\mathbf{T};\quad \frac{d\mathbf{P}'}{dt}=\boldsymbol{\xi}\times\mathbf{P}';\quad \frac{d\mathbf{Q}'}{dt}=\boldsymbol{\xi}\times\mathbf{Q}'. \tag{4.27}$$

Here $\boldsymbol{\xi}$ is given by,

$$\boldsymbol{\xi}=\alpha_3\mathbf{T}+\alpha_1\mathbf{Q}'-\alpha_2\mathbf{P}'. \tag{4.28}$$

Equations (4.27) show that the natural frame $(\mathbf{T},\mathbf{P}',\mathbf{Q}')$ rotates with an angular velocity $\boldsymbol{\xi}$, as it moves along the space curve. As is obvious, α_1 and α_2 are components of $\boldsymbol{\xi}$ along the $\mathbf{Q}'$ and $\mathbf{P}'$ axes respectively and hence *tilt* the $(\mathbf{P}',\mathbf{Q}')$ plane. On the other hand, α_3 merely *rotates* this plane around $\mathbf{T}$. Thus in time T, the $(\mathbf{P}',\mathbf{Q}')$ plane gets rotated by an angle $\gamma_d=\int_0^T\alpha_3 dt$. Such a frame is defined using Fermi-Walker parallel transport as [21],

$$\frac{DA^i}{dt}=\{(\alpha_1\mathbf{Q}'-\alpha_2\mathbf{P}')\times A\}^i.$$

Using the expression for α_3 given in (4.26) we obtain the dynamical phase γ_d associated with $(\mathbf{P}',\mathbf{Q}')$ plane to be

$$\gamma_d=\int_0^T\alpha_3 dt=-2\int_0^T\left(\sin^2\frac{\alpha}{2}\right)\frac{d\phi}{dt}\,dt-2(\theta_1(T)-\theta_1(0)), \tag{4.29}$$

since $\beta=2\theta_1$, from equation (4.18).

Subtracting equation (4.29) from the expression for the total phase γ_p given in equation (4.11) we obtain the geometric phase γ_g associated with $(\mathbf{P}',\mathbf{Q}')$ rotation to be

$$\gamma_g=\gamma_p-\gamma_d=2\left[\int_0^T\left(\sin^2\frac{\alpha}{2}\right)\frac{d\phi}{dt}\;dt-\Delta\right]. \tag{4.30}$$

Note that the term involving the gauge freedom β cancels out here too, as in the kinematic approach. Comparing equation (4.30) with the expression for ϕ_g, we see that

$$\gamma_g=-2\phi_g.$$

In other words, the geometric phase ϕ_g associated with the wave function is minus half of that associated with the $(\mathbf{P}',\mathbf{Q}')$ rotation. Note that equation (4.30) is valid for *non-adiabatic* as well as *non-cyclic* evolutions. For a cyclic evolution, $\Delta=0$. Here, on computing γ_g, the geometric phase ϕ_g becomes just minus half the solid angle, as is well known. In summary, by mapping the evolution equation for the wavefunction to the dynamical equation for an orthonormal triad $(\mathbf{T},\mathbf{P}',\mathbf{Q}')$ and identifying the triad to be a natural frame on a space curve, enables us to provide a purely geometrical visualisation of the geometric phase of a two level system.

Our general result is valid for any two level system with the wavefunction (4.1), since we did not use the specific BJJ equations (4.2) and (4.3) in its derivation. By finding the solutions $\alpha(t)$ and $\phi(t)$ for the nonlinear equations numerically for given initial conditions, γ_g can be computed and is exactly $-2\phi_g$, with ϕ_g values as plotted in Figures 6 and 7.

5 Geometric parameters associated with BJJ dynamics

In the last section, we discussed the mapping of the BJJ tunneling equations to a space curve which is described using equations for a "natural frame". This description involves three geometrical parameters α_i which are shown to depend on a gauge parameter β (see Eqs. (4.20) and (4.24)).

The usual description of a space curve is in terms of a Frenet frame [20], with the curvature K and torsion τ as the geometric parameters. As explained in Section 4, working with the Frenet frame implies *fixing* $\beta=\beta_F$, defined in equation (4.23). In this section we work with the Frenet frame to determine the geometric parameters K and τ of the space curve associated with the BJJ dynamics, in terms of the physical parameters $V,\Delta E$ and Λ and discuss certain special cases of interest.

As mentioned in Section 4, in the Frenet frame, $\alpha_1=K,\alpha_2=0,\ \ \alpha_3=\tau,\ \ \mathbf{P}=\mathbf{n}$ and $\mathbf{Q}=\mathbf{b}$ in equations (4.15) to (4.17). In this frame, we have the usual Frenet-Serret equations [20],

$$\frac{d\mathbf{T}}{dt}=K\mathbf{n},\quad \frac{d\mathbf{n}}{dt}=-K\mathbf{T}+\tau\mathbf{b};\quad \frac{d\mathbf{b}}{dt}=-\tau\mathbf{n}. \tag{5.1a}$$

Thus,

$$\frac{d\mathbf{T}}{dt}=\boldsymbol{\xi}_F\times\mathbf{T};\quad \frac{d\mathbf{n}}{dt}=\boldsymbol{\xi}_F\times\mathbf{n};\quad \frac{d\mathbf{b}}{dt}=\boldsymbol{\xi}_F\times\mathbf{b}. \tag{5.1b}$$

Also,

$$K^2=\left(\frac{d\mathbf{T}}{dt}\right)^2=\ \sin^2\alpha\left(\frac{d\phi}{dt}\right)^2+\left(\frac{d\alpha}{dt}\right)^2 \tag{5.2}$$

and

$$\tau=\mathbf{T}\cdot(\dot{\mathbf{T}}\times\ddot{\mathbf{T}})/K^2=\cos\alpha\frac{d\phi}{dt}+\frac{d}{dt}\tan^{-1}\left(\frac{\sin\alpha\frac{d\phi}{dt}}{\frac{d\alpha}{dt}}\right), \tag{5.3}$$

where the Cartesian representation (3.8), $\mathbf{T} = \mathbf{r}$ has been used. On using expressions for $d\alpha/dt$, $d\phi/dt$ given in equations (2.9), in equations (5.2) and (5.3) respectively, we get

$$K = 2(V^2 + \hbar^2\omega_0^2 \sin^2\alpha - V^2 \sin^2\alpha \cos^2\phi + 2V\hbar\omega_0 \cos\alpha \sin\alpha \cos\phi)^{1/2}, \quad (5.4)$$

and

$$\tau = \cos\alpha(\hbar\omega_0 + V\cot\alpha\cos\phi) + \frac{d}{dt}\tan^{-1}\left(\frac{\hbar\omega_0 \sin\alpha + V\cos\alpha\cos\phi}{V\sin\phi}\right). \quad (5.5)$$

Equations (5.4) and (5.5) give the curvature and torsion of the space curve created by the BJJ dynamics with parameters V and $\hbar\omega_0$, which in turn are defined in terms of the condensate parameters.

Since $\mathbf{r}$ is identified with $\mathbf{T}$, we also have, for the BJJ system,

$$\frac{d\mathbf{T}}{dt} = \boldsymbol{\omega} \times \mathbf{T}, \quad \boldsymbol{\omega} = (-2V, 0, 2\omega_0). \quad (5.6)$$

From equation (4.2) therefore K^2 can also be written as,

$$K^2 = \left(\frac{d\mathbf{T}}{dt}\right)^2 = (\boldsymbol{\omega})^2 - (\boldsymbol{\omega}\cdot\mathbf{T})^2. \quad (5.7)$$

Using the definition of M_{ω_0} given in equation (2.3), a simple calculation shows that $2\langle M_{\omega_0}\rangle = \boldsymbol{\omega}\cdot\mathbf{T}$, yielding

$$4\langle M_{\omega_0}^2\rangle = (\boldsymbol{\omega})^2. \quad (5.8)$$

Using equation (5.8) in equation (5.7), we get

$$K = 2(\langle M_{\omega_0}^2\rangle - \langle M_{\omega_0}\rangle^2)^{1/2}. \quad (5.9)$$

Now from the first equation in equation (5.1) it is clear that the distance traveled by the tip of $\mathbf{T}$ on the unit sphere in time dt is $ds = K dt$.This is the well-known [22] Fubini-Study metric. Thus we see that the curvature K which determines the geometric quantity ds is given by the variance of the tunneling matrix M_{ω_0} for a two level system. As seen from the equation (5.3), the torsion integral $\int \tau dt$ measures the anholonomy of the frame, i.e. a path dependent geometric quantity given by the solid angle associated with a cyclic evolution of $\mathbf{T}$.

Recalling that the population density difference between the two traps is given by z and the phase difference by ϕ, it is instructive to write the geometric quantities K and τ in terms of these physical quantities and the system parameters $V, \Delta E$ and Λ: from equation (5.4),

$$K = 2(V^2 + (\Delta E + \Lambda z)^2(1 - z^2) - V^2(1 - z^2)\cos^2\phi + 2V(\Delta E + \Lambda z)z\sqrt{1 - z^2}\cos\phi)^{1/2}. \quad (5.10)$$

After a short calculation K can be written as,

$$K = 2\left[V^2 + (\Delta E + \Lambda z)^2 - \left(H_{cl} + \frac{\Lambda z^2}{2}\right)^2\right]^{1/2}, \quad (5.11)$$

where H_{cl} is the effective classical Hamiltonian given in equation (2.7), which leads to the integrable dynamics of z and ϕ. Next from equation (5.5) we obtain τ:

$$\tau = z\left(\Delta E + \Lambda z + \frac{Vz}{\sqrt{1 - z^2}}\cos\phi\right) + \frac{d}{dt}\tan^{-1}\frac{(\Delta E + \Lambda z)\sqrt{1 - z^2} + Vz\cos\phi}{V\sin\phi}. \quad (5.12)$$

Using the expression for H_{cl} once again, we get

$$\tau = H_{cl} + \frac{\Lambda z^2}{2} + \frac{V}{\sqrt{1 - z^2}}\cos\phi + \frac{d}{dt}\tan^{-1}\frac{(\Delta E + \Lambda z)\sqrt{1 - z^2} + Vz\cos\phi}{V\sin\phi}. \quad (5.13)$$

We consider some special cases.

(1) Interacting Bose system with no external potential: ($V = 0$, $\Lambda \neq 0$). From equation (2.6a), setting $V = 0$, we get $z =$ constant. This in turn yields $\tau = H_{cl} + \Lambda z^2/2 =$ constant and $K = 2(\Delta E + \Lambda z)\sqrt{1 - z^2} =$ constant i.e., the underlying geometry is that of a circular helix with a constant pitch.

(2) The ideal Bose gas in an external potential: ($\Lambda = 0$, $V \neq 0$). If one considers a non-interacting Bose system then setting $\Lambda = 0$ in equation (2.7), we get

$$H_{cl} = -V\sqrt{1 - z^2}\cos\phi + \Delta E z. \quad (5.14)$$

We note that the tunneling Hamiltonian resembles that of a two-component BEC in the rotating frame approximation [25]. From equation (5.13), we see that,

$$\tau = H_{cl} + \frac{V}{\sqrt{1 - z^2}}\cos\phi + \frac{d}{dt}\tan^{-1}\frac{\Delta E\sqrt{1 - z^2} + Vz\cos\phi}{V\sin\phi}. \quad (5.15)$$

Further, from equation (5.11),

$$K = 2(V^2 + (\Delta E)^2 - H_{cl})^{1/2}. \quad (5.16)$$

Since H_{cl} is a constant under time evolution, equation (5.16) shows that the curvature K is a constant. However, the torsion τ is time-dependent in this case. Since K is a constant, the path length on the unit sphere as given by the Fubini-Study metric is linearly dependent on time for this case. (If V and ΔE are made time dependent, then K is not a constant any more.)

(3) The linear limit. For a symmetric trap with $\Delta E = 0$ in the small oscillations limit, linearizing equations (2.6) in *both* z and ϕ, for $|z| \ll 1$, $|\phi| \ll 1$, we get,

$$\frac{dz}{dt} = -V\phi, \quad \frac{d\phi}{dt} = (\Lambda + V)z, \quad (5.17)$$

and

$$H = (\Lambda + V)\frac{z^2}{2} + V\frac{\phi^2}{2}.$$

This is just the harmonic oscillator limit and analytical solutions are known. The corresponding expressions for K and τ can be calculated using equations (5.11) and (5.12).

(4) The rigid pendulum limit. For a symmetric trap with $\Delta E = 0$, linearizing equations (2.3) in z only, with $\Lambda \gg 1$ we get the equations of a pendulum with fixed length,

$$\frac{dz}{dt} = -V \sin\phi \quad \frac{d\phi}{dt} = \Lambda z. \tag{5.18}$$

As is well-known the solutions for z can be written in terms of Elliptic functions thus K and τ can be obtained from equations (5.11) and (5.12).

Finally, for a symmetric trap $\Delta E = 0$, with no linearizing approximations, though the analytical solution of equations (2.6) can be found [12], it is easier to work with numerical solutions instead, using which K and τ can be computed numerically using the expressions (5.11) and (5.12).

6 Summary of results and possible experiments

The geometric phase associated with the time evolution of the wave function of a Bose-Einstein condensate in a double well trap has been found using a quantum approach. We have explicitly computed the geometric phase ϕ_g for both cyclic and noncyclic evolutions of the condensate population density difference z and phase difference ϕ in the two wells, by taking an example. We have shown that the geometric phase can also be derived using a classical differential geometric approach, by essentially mapping the evolution of the two states to a framed space curve with natural moving frames along the curve. The unit tangent vector $\mathbf{r}$ to the curve has $\alpha = \cos^{-1} z$ as the polar angle and ϕ as the azimuthal angle. As we have shown, here the geometric phase arises due to the path-dependent rotation of the frame perpendicular to $\mathbf{r}$ as the system evolves in time.

It should be noted that $\mathbf{r}$ defined in equation (4.4) has a (classical) angular momentum form in the sense that its components are the classical counterparts of the Schwinger two-boson realization of the angular-momentum picture employed by some authors [23]. This could lead to some useful links between these papers and our work.

We remark that in a recent paper, Liu et al. [24] have obtained some interesting results in the adiabatic theory of nonlinear evolution of quantum states. By applying their elegant analysis to the example of a tunneling model of a Bose-Einstein condensate, they have shown that when the non-eigenstates are evolved *adiabatically*, the Aharonov-Anandan phases play the role of classical canonical actions. In comparison, our work is concerned with *nonadiabatic* evolution of non-eigenstates. Further, our methodology is based on the study of the geometry of the system by mapping it to a space curve with an associated frame-field, and is quite distinct from theirs.

In an experimental set up, suppose one designs a double-well trap by creating a barrier within a trapped condensate with N atoms, using a laser sheet. We propose that in an actual experiment, immediately after creating the laser sheet barrier, if the density difference and the phase difference between the condensates in the two traps can be measured as a function of time, then by substituting these experimentally measured functions in equation (3.9), the associated geometric phase ϕ_g can be determined. As ϕ_g will depend on system parameters as well as initial conditions, this experiment would enable one to study the variation of ϕ_g with trap parameters, which would be useful in designing appropriate experiments to measure it.

Another type of experiment to study tunneling between condensates, proposed by Williams et al. [25], hinges on the fact that it has become possible to confine a two-component Bose condensate in the same trap, as follows. Hall et al. [26] first trapped and cooled ^{87}Rb atoms in a magnetic trap in the $|f = 1, m_f = -1\rangle$ hyperfine state. After condensation, it is possible to populate the $|f = 2, m_f = 1\rangle$ hyperfine state through a two-photon transition. In the presence of a weak magnetic field, these states are separated in energy by ω_o (say). Thus two different hyperfine states can exist in the trap. A weak two-photon driving pulse is applied which couples the two states and consequently, atoms can “tunnel” between the two condensates. In this model, it has been shown [25] that in the mean field approximation, one obtains coupled equations for $z(t)$ and $\phi(t)$ almost identical in form to equations (2.7), but with $\Lambda = 0$ (i.e., non-interacting) with the other parameters appropriately defined for the model. Hence all our results for the geometric phase are applicable here as well.

Recently, Fuentes-Guirdi et al. [27] have proposed a method for generating a geometric phase in a coupled two-mode Bose Einstein condensate, starting with a Hamiltonian for two condensates existing in *different* hyperfine states. In addition to the experiments of Hall mentioned above, condensates of ^{87}Rb atoms in hyperfine states $|f = 1, m_f = 1\rangle$ and $|f = 2, m_f = 2\rangle$ have been produced experimentally [28]. Likewise, condensates of ^{23}Na atoms with $|f = 1, m_f = 1\rangle$ and $|f = 1, m_f = 0\rangle$ have also been created [29]. Using the Schwinger angular momentum representation, the Hamiltonian describing two coupled hyperfine states $|A\rangle$ and $|B\rangle$ can be expressed in the form [27]

$$H_{hf} = \alpha_0 J_z + \beta_0 J_z^2 + \gamma_0 [J_x \cos\phi_D + J_y \sin\phi_D]. \tag{6.1}$$

Here, (J_x, J_y, J_z) are the components of an effective ‘mesoscopic’ spin $\mathbf{J}$, since it can be shown that J is proportional to the total number of atoms N in the condensate, which is of the order of 10^4. In equation (6.1), $\phi_D = D\,t$, D being the detuning frequency of the laser which couples the two hyperfine states. Further, α_0 and β_0 are system parameters and γ_0 is the strength of the laser-induced drive term that couples the two levels.

Interestingly, if we write the components of $\mathbf{J}$ in the form $\mathbf{J} = (J_x, J_y, J_z) = J(\sin\alpha\cos\phi, \sin\alpha\sin\phi, \cos\alpha)$ in equation (6.1), then on setting $\phi_D = 0$, H_{hf}/J becomes identical to our Hamiltonian in equation (2.7), on identifying $\alpha_0 = \Delta E$, $\beta_0 = \Lambda/2$ and $\gamma_0 = -V$. Conversely, if an external driving field phase ϕ_D is subtracted from ϕ in equation (2.10), we would essentially obtain equation (6.1). Thus our results for the geometric phase will be valid for that case too, with the appropriate parameters substituted.

It would be of interest to measure geometric phase more directly, for the BJJ. In the context of superconductor Josephson Junctions, the geometric phases for adiabatic [30,31] and nonadiabatic evolutions [32] have been studied. In analogy with that discussion, we write our Hamiltonian (2.7) as $H = (-1/2)\mathbf{B}\cdot\boldsymbol{\sigma}$, where for the noninteracting case, our fictitious magnetic field is $\mathbf{B} = (-V, 0, \Delta E)$. By tuning the laser sheet parameters appropriately, we can take the system through a cyclic path on the Bloch sphere (see Fig. 2 in Ref. [32]), the path being made up of geodesic curves which have *vanishing* dynamic phase. If the interference pattern is then studied, the measured relative phase is therefore just the geometric phase. The effect of noise on geometric phases has been discussed in [33].

Experimental techniques to create two condensates in close proximity have been suggested recently [34] in the context of producing a continuous source of a Bose-Einstein condensate. It would be interesting to study the tunneling between the condensates in such a set up, if feasible.

Geometric phases have been recently shown to have relevance in the implementation of fault-tolerant quantum computation [35,36], and in the creation of vortices in a condensate [37]. We hope that our results will have applications in these contexts as well.

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